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Ground-state structure and low temperature behaviour of an integrable chain with alternating spins

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Abstract. In this paper we continue the investigation of an anisotropic integrable spin chain, consisting of spins $s = 1$ and $s = \frac{1}{2}$, started in our paper [1]. The thermodynamic Bethe ansatz is analysed especially for the case, when the signs of the two couplings $\bar{c}$ and $\tilde{c}$ differ. For the conformally invariant model ($\bar{c} = \tilde{c}$) we have calculated heat capacity and magnetic susceptibility at low temperature. In the isotropic limit our analysis is carried out further and susceptibilities are calculated near phase transition lines (at $T = 0$).

1. Introduction

Since the pioneering work of de Vega and Woynarovich [2] for the construction of models with alternating spins quite a lot of interesting generalizations have been presented [3–5]. Otherwise, not many results were obtained concerning the physical structure of the models, e.g. the low temperature behaviour of heat capacity and magnetic susceptibility. Even the structure of the ground state in the framework of the Bethe ansatz is not fully understood for the original model.

In this paper, therefore, we continue our investigation of the $XXZ(\frac{1}{2}, 1)$ model with strictly alternating spins started in [1], which will be referred to as paper I throughout.

In section 3 the thermodynamic Bethe ansatz (TBA) is analysed for zero temperature in different regions of coupling constants. Section 4 deals with the conformally invariant model, where the low temperature behaviour can be determined analytically. In section 5 we derive some new results for the isotropic case $XXX(\frac{1}{2}, 1)$. Our conclusions are contained in section 6.

2. Definition of the model

We refer the reader to paper I and [2] for the basics of the model; we will follow the definitions and notations of paper I here.

Our Hamiltonian of a spin chain of length $2N$ is given by

$$\mathcal{H}(\gamma) = \bar{c}\bar{\mathcal{H}}(\gamma) + \tilde{c}\tilde{\mathcal{H}}(\gamma) - HS^z \tag{2.1}$$

with the two real coupling constants $\bar{c}$ and $\tilde{c}$. The anisotropy parameter γ is limited to $0 < \gamma < \pi/2$.

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For convenience we repeat the Bethe ansatz equations (BAE) and the magnon energies:

$$\left(\frac{\sinh(\lambda_j + \mathrm{i}\frac{\gamma}{2})}{\sinh(\lambda_j - \mathrm{i}\frac{\gamma}{2})}\frac{\sinh(\lambda_j + \mathrm{i}\gamma)}{\sinh(\lambda_j - \mathrm{i}\gamma)}\right)^N = -\prod_{k=1}^{M}\frac{\sinh(\lambda_j - \lambda_k + \mathrm{i}\gamma)}{\sinh(\lambda_j - \lambda_k - \mathrm{i}\gamma)} \qquad j = 1\ldots M \tag{2.2}$$

$$E = \bar{c}\bar{E} + \tilde{c}\tilde{E} - \left(\frac{3N}{2} - M\right)H \tag{2.3}$$

$$\bar{E} = -\sum_{j=1}^{M}\frac{2\sin\gamma}{\cosh 2\lambda_j - \cos\gamma} \tag{2.4}$$

$$\tilde{E} = -\sum_{j=1}^{M}\frac{2\sin 2\gamma}{\cosh 2\lambda_j - \cos 2\gamma}. \tag{2.5}$$

3. Thermodynamic Bethe ansatz and the ground state for different signs of the coupling constants

In section 3 of paper I the TBA was considered for special values of $\gamma = \pi/\mu$, μ integer and $\mu \geqslant 3$. We also argued, that the ground state structure is uniform in our whole γ-region, while we expect possible changes for the excitations at the γ-points above. We therefore use the results of paper I for the possible appearance of strings in the ground state according to the different regions of couplings.

For completeness we quote in all cases of the TBA, equations (3.19) of paper I. We found it more convenient to use λ-space instead of Fourier transformation, which can be easily derived from our equations below. We then recall

$$f'(\lambda, n, \pm 1) = \pm\frac{2\sin n\gamma}{\cosh 2\lambda \mp \cos n\gamma}. \tag{3.1}$$

For shortness we drop the magnetic field in the TBA, it can be added later without any problem.

Now we analyse the zero-temperature TBA in the various regions of signs for $\bar{c}$ and $\tilde{c}$.

(i) $\bar{c} > 0$, $\tilde{c} > 0$.

$$\begin{aligned}\epsilon_1^+(\lambda) = {} & -\bar{c}f'(\lambda,1,1) - \tilde{c}f'(\lambda,2,1) - \left[\delta(\lambda) + \frac{f'(\lambda,2,1)}{2\pi}\right] * \epsilon_1^- \\ & - \left[\frac{f'(\lambda,1,1) + f'(\lambda,3,1)}{2\pi}\right] * \epsilon_2^-\end{aligned} \tag{3.2}$$

$$\begin{aligned}\epsilon_2^+(\lambda) = {} & -\bar{c}f'(\lambda,2,1) - \tilde{c}[f'(\lambda,1,1) + f'(\lambda,3,1)] - \left[\frac{f'(\lambda,1,1) + f'(\lambda,3,1)}{2\pi}\right] * \epsilon_1^- \\ & - \left[\delta(\lambda) + \frac{2f'(\lambda,2,1) + f'(\lambda,4,1)}{2\pi}\right] * \epsilon_2^-\end{aligned} \tag{3.3}$$

with the convolution $a * b(\lambda)$ defined as

$$a * b(\lambda) = \int_{-\infty}^{\infty} a(\lambda - \mu)b(\mu)\,\mathrm{d}\mu. \tag{3.4}$$

The solution has already been given in [2], where the excitations have also been found.

(ii) $\bar{c} > 0$, $\tilde{c} < 0$.

We expect $(1,+)$ and $(1,-)$ strings

$$\epsilon_1^+(\lambda) = -\bar{c}f'(\lambda,1,1) - \tilde{c}f'(\lambda,2,1) - \left[\delta(\lambda) + \frac{f'(\lambda,2,1)}{2\pi}\right] * \epsilon_1^-$$

$$+\left[\frac{f'(\lambda,2,-1)}{2\pi}\right]*\epsilon_{-1}^{-} \tag{3.5}$$

$$\epsilon_{-1}^{+}(\lambda) = -\bar{c}f'(\lambda,1,-1) - \tilde{c}f'(\lambda,2,-1) - \left[\frac{f'(\lambda,2,-1)}{2\pi}\right]*\epsilon_{1}^{-} - \left[\delta(\lambda) - \frac{f'(\lambda,2,1)}{2\pi}\right]*\epsilon_{-1}^{-}. \tag{3.6}$$

At first it might be expected that the solution is given when both strings are distributed with infinite Fermi radius. We have determined this state and calculated its energy, but it is not the ground state. The same applies to the state with only $(1,+)$ strings. That can already be seen superficially after obtaining $S_z \neq 0$ for it.

The situation changes when only $(1,-)$ strings are considered. This is due to the fact, that the last two terms in equation (3.5) are definitely non-negative while this is not the case in equation (3.6), where the term after the δ-function spoils the argument.

Equation (3.6) for $\epsilon_1^-(\lambda) \equiv 0$ has been already solved in paper I.

$$\epsilon_{-1}^{-}(\lambda) = \frac{\pi\bar{c}}{\pi-\gamma}\frac{1}{\cosh(\pi\lambda/(\pi-\gamma))} + \frac{4\pi\tilde{c}}{\pi-\gamma}\frac{\cos(\pi\gamma/2(\pi-\gamma))\cosh(\pi\lambda/(\pi-\gamma))}{\cosh(2\pi\lambda/(\pi-\gamma))+\cos(\pi\gamma/(\pi-\gamma))}. \tag{3.7}$$

Introducing the function $g(\lambda,\alpha)$

$$g(\lambda,\alpha) = \frac{4\pi}{\pi-\gamma}\frac{\cos(\pi\alpha/2(\pi-\gamma))\cosh(\pi\lambda/(\pi-\gamma))}{\cosh(2\pi\lambda/(\pi-\gamma))+\cos(\pi\alpha/(\pi-\gamma))} \tag{3.8}$$

the solution of equation (3.5) can be written as

$$\epsilon_1^{+}(\lambda) = -\bar{c}g(\lambda,\pi/2-\gamma) - \tilde{c}g(\lambda,\pi/2-3\gamma/2). \tag{3.9}$$

For consistency it is necessary to have

$$\epsilon_{-1}^{-}(\lambda) \leqslant 0 \qquad \text{and} \qquad \epsilon_1^{+}(\lambda) \geqslant 0. \tag{3.10}$$

Both conditions specify the region of $\bar{c}$ and $\tilde{c}$ where our solution is valid.

We start with $\epsilon_1(\lambda)$

$$\epsilon_1(0) = -\frac{2\pi}{\pi-\gamma}\left[\frac{\bar{c}}{\cos(\pi(\pi-2\gamma)/2(\pi-\gamma))} + \frac{\tilde{c}}{\cos(\pi(\pi-3\gamma)/2(\pi-\gamma))}\right] \geqslant 0. \tag{3.11}$$

Considering the asymptotics for $\lambda \to \infty$ one has

$$-\frac{2\pi}{\pi-\gamma}\left[\bar{c}\cos\frac{\pi(\pi-2\gamma)}{2(\pi-\gamma)} + \tilde{c}\cos\frac{\pi(\pi-3\gamma)}{2(\pi-\gamma)}\right] \geqslant 0. \tag{3.12}$$

We now assume that the two necessary conditions (3.11) and (3.12) are also sufficient to fulfill the second part of (3.10).

The smaller of the ratios of the two cosine functions is then the upper limit of $\bar{c}/|\tilde{c}|$. Hence after elementary recasting

$$\begin{aligned} \frac{\bar{c}}{|\tilde{c}|} &\leqslant \frac{1}{2\cos(\pi\gamma/2(\pi-\gamma))} \qquad 0 \leqslant \gamma \leqslant \frac{2\pi}{5} \\ \frac{\bar{c}}{|\tilde{c}|} &\leqslant 2\cos\frac{\pi\gamma}{2(\pi-\gamma)} \qquad \frac{2\pi}{5} \leqslant \gamma < \frac{\pi}{2}. \end{aligned} \tag{3.13}$$

We treat $\epsilon_{-1}^{-}(\lambda)$ in the same way obtaining

$$\frac{\bar{c}}{|\tilde{c}|} \leqslant 2\cos\frac{\pi\gamma}{2(\pi-\gamma)}. \tag{3.14}$$

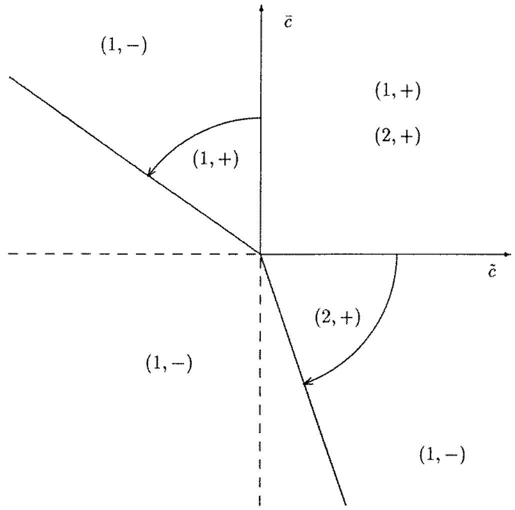

Figure 1. The phase structure of the $XXZ(\frac{1}{2}, 1)$ model for $\gamma = \pi/3$. The ground state strings are indicated for the four different sectors. The arrows symbolize the decreasing Fermi radii of the corresponding strings. Broken lines are coordinate axes. Where axes are drawn full they coincide with sector borders.

Now it is not difficult to show that condition (3.14) is fulfilled, when (3.13) holds.

Therefore, our solution, a sea of $(1, -)$ strings with infinite Fermi zone, is the ground-state configuration as long as the inequalities (3.13) hold. In the $(\tilde{c}, \bar{c})$-plane this is an open triangle formed by the negative $\tilde{c}$-axis and the straight line given by relation (3.13) when the equality holds (see figure 1). For $\gamma \to 0$ (isotropic case, see section 5) this is $\bar{c}/|\tilde{c}| = \frac{1}{2}$. For increasing γ the region first enlarges until $\gamma = 2\pi/5$ and then shrinks and approaches the $\tilde{c}$-axis when $\gamma \to \pi/2$.

Above that line we still expect $(1, -)$ strings but together with $(1, +)$ strings. So moving counter-clockwise from the positive $\bar{c}$-axis towards that line the Fermi radius of the strings with positive parity shrinks from infinity to zero, while the radius for the strings with negative parity is infinite, as can easily be seen from paper I (3.17), which implies in the case $H = 0$, that its energy function does not change sign and is therefore strictly non-positive in the limit $T \to 0$.

It is remarkable that a finite Fermi zone occurs without the presence of a magnetic field. Apparently the second coupling plays the role of an external field.

(iii) $\bar{c} < 0$, $\tilde{c} > 0$.

We expect $(2, +)$ and $(1, -)$ strings

$$
\begin{aligned}
\epsilon_2^+(\lambda) = -\bar{c} f'(\lambda, 2, 1) - \tilde{c}[f'(\lambda, 1, 1) + f'(\lambda, 3, 1)] \\
- \left[\delta(\lambda) + \frac{2f'(\lambda, 2, 1) + f'(\lambda, 4, 1)}{2\pi}\right] * \epsilon_2^- \\
+ \left[\frac{f'(\lambda, 1, -1) + f'(\lambda, 3, -1)}{2\pi}\right] * \epsilon_{-1}^- \qquad (3.15) \\
\epsilon_{-1}^+(\lambda) = -\bar{c} f'(\lambda, 1, -1) - \tilde{c} f'(\lambda, 2, -1) - \left[\frac{f'(\lambda, 1, -1) + f'(\lambda, 3, -1)}{2\pi}\right] * \epsilon_2^-
\end{aligned}
$$

$$-\left[\delta(\lambda) - \frac{f'(\lambda, 2, -1)}{2\pi}\right] * \epsilon_{-1}^{-}. \tag{3.16}$$

We have found that qualitatively the same arguments apply as in case (ii) above. Thus, we consider first only $(1, -)$ strings with infinite Fermi radius. Now it is necessary to assure $\epsilon_2^+(\lambda) \geqslant 0$ in addition to the first condition of (3.10). Instead of condition (3.14) it gives now

$$\frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{2}{\cos(\pi\gamma/2(\pi - \gamma))} \tag{3.17}$$

which guarantees $\epsilon_{-1}^{-}(\lambda) \leqslant 0$. When calculating ϵ_2^+ one has to be careful when the Fourier transformation of $f'(\lambda, 3, -1)$ is to be taken. It vanishes for $\gamma = \pi/3$ and changes the sign after that point has been passed. Finally one obtains

$$\begin{aligned} &\epsilon_2^+(\lambda) = -\bar{c}g(\lambda, \pi/2 - 3\gamma/2) - \tilde{c}g(\lambda, \pi/2 - \gamma) - \tilde{c}g(\lambda, \pi/2 - 2\gamma) \\ &\qquad 0 < \gamma < \pi/3 \\ &\epsilon_2^+(\lambda) \equiv 0 \qquad \pi/3 < \gamma < \pi/2. \end{aligned} \tag{3.18}$$

There is no contradiction with paper I (3.24), which gives two different values for $\gamma < \pi/3$ and $\gamma = \pi/3$, while larger γ-values were not considered there.

Let us first consider $0 < \gamma < \pi/3$. Then from equation (3.17) we have the two conditions

$$\begin{aligned} &-\frac{2\pi}{\pi - \gamma}\left(\frac{\bar{c}}{\cos(\pi(\pi - 3\gamma)/2(\pi - \gamma))} + \frac{\tilde{c}}{\cos(\pi(\pi - 2\gamma)/2(\pi - \gamma))}\right. \\ &\qquad \left. + \frac{\tilde{c}}{\cos(\pi(\pi - 4\gamma)/2(\pi - \gamma))}\right) \geqslant 0 \\ &-\frac{2\pi}{\pi - \gamma}\left(\bar{c}\cos\frac{\pi(\pi - 3\gamma)}{2(\pi - \gamma)} + \tilde{c}\cos\frac{\pi(\pi - 2\gamma)}{2(\pi - \gamma)} + \tilde{c}\cos\frac{\pi(\pi - 4\gamma)}{2(\pi - \gamma)}\right) \geqslant 0. \end{aligned} \tag{3.19}$$

Straightforward calculation gives

$$\frac{|\bar{c}|}{\tilde{c}} \geqslant 2\cos\frac{\pi\gamma}{2(\pi - \gamma)} \qquad \text{and} \qquad \frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{8\cos^3(\pi\gamma/2(\pi - \gamma))}{4\cos^2(\pi\gamma/2(\pi - \gamma)) - 1}. \tag{3.20}$$

The upper term of the r.h.s. is always smaller than the r.h.s. of (3.17). Hence we have to find the maximum of the two r.h.s. of formula (3.20) and (3.17). In our γ-region the second inequality of (3.20) is the most restrictive one. Putting things together we find for the region with $(1, -)$ strings only

$$\begin{aligned} &\frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{8\cos^3(\pi\gamma/2(\pi - \gamma))}{4\cos^2(\pi\gamma/2(\pi - \gamma)) - 1} \qquad 0 < \gamma \leqslant \frac{\pi}{3} \\ &\frac{|\bar{c}|}{\tilde{c}} \geqslant \frac{2}{\cos\pi\gamma/2(\pi - \gamma)} \qquad \frac{\pi}{3} \leqslant \gamma < \frac{\pi}{2}. \end{aligned} \tag{3.21}$$

The $(\tilde{c}, \bar{c})$-plane is an open triangle formed by the negative $\bar{c}$-axis and the straight line given by relation (3.21) when the equality holds (see figure 1). For $\gamma \to 0$ (isotropic case, see section 5) this is $|\bar{c}|/\tilde{c} = \frac{8}{3}$. For rising γ the region shrinks and approaches the $\tilde{c}$-axis when $\gamma \to \pi/2$.

Above that region we expect $(1, -)$ strings together with $(2, +)$ strings the latter with finite Fermi radius. The picture resembles region (ii) above.

(iv) $\bar{c} \leqslant 0$, $\tilde{c} \leqslant 0$.

Here the vacuum is formed by $(1,-)$ strings only.

$$\epsilon^{+}_{-1}(\lambda) = -\bar{c} f'(\lambda,1,-1) - \tilde{c} f'(\lambda,2,-1) - \left[\delta(\lambda) + \frac{f'(\lambda,2,-1)}{2\pi}\right] * \epsilon^{-}_{-1}. \tag{3.22}$$

This region was studied in paper I where the excitations have also been found.

(v) $\bar{c}=0$, $\tilde{c}>0$.

(vi) $\bar{c}>0$, $\tilde{c}=0$.

We add nothing new to both cases considered earlier in [2] and [6].

Now we can summarize our results about the the ground state structure for different values of coupling constants. There are four regions and two singular lines (v) and (vi). In the two regions with equal signs (which contain the line $\bar{c}=\tilde{c}$) the ground state is independent of the values of $\bar{c}$ and $\tilde{c}$. Here also the Fermi radii are infinite. There is no mass gap in the excitation spectrum.

In the two other regions infinite and finite Fermi radii occur and the concrete structure of the ground state depends on the ratio $\bar{c}/\tilde{c}$. Nevertheless, we expect them to be gapless also.

The picture is not fully symmetric, because region (i) is separated from all others by a highly degenerate ground state on both lines. This is connected with the fact that one sort of string has to disappear at once.

Finally, the model shows an antiferromagnetic behaviour everywhere (for vanishing magnetic field) as long as $\gamma>0$. The isotropic case is considered in section 5.

4. Calculation of the low-temperature behaviour in the case $\bar{c}=\tilde{c}$

In this section we calculate the low-temperature heat capacity and magnetic susceptibility for vanishing magnetic field in the case $\bar{c}=\tilde{c}$.

We therefore go back to equations (3.10)–(3.13) of paper I where T is considered to be small but finite. Instead of [6] where the free energy was calculated, we use a method due to Wiegmann [8], which for our purpose was used by Babujian and Tsvelick [9] to obtain the results for the $XXZ(S)$ model from entropy and polarization. To explore this method it is necessary to ensure $\gamma<\pi/3$, $\gamma=\pi/\mu$; μ integer.

We will present in some detail the case $c=\bar{c}=\tilde{c}<0$ while for $c>0$ we mention only the necessary changes and the final results.

For $c<0$ we have

$$\begin{aligned} &\epsilon_j \geqslant 0 \qquad j=1\ldots\mu-1 \\ &\epsilon_\mu \leqslant 0 \end{aligned}$$

and

$$\begin{aligned} &\rho_j \to 0 \qquad \text{for } T\to 0 \qquad \text{if } j=1\ldots\mu-1 \\ &\tilde{\rho}_\mu \to 0 \qquad \text{for } T\to 0. \end{aligned}$$

Our aim is now to recast BAE and TBAE in a form where energies and densities are given through their zero temperature limits $\epsilon^{(0)}_\mu$ and $\epsilon^{(0)}_j$ and values vanishing for $T\to 0$, i.e. the energy functions $\ln(1+\exp(-\epsilon_j/T))$, $\quad j=1\ldots\mu-1$ and $\ln(1+\exp(\epsilon_\mu/T))$.

The main step is the multiplication by $A_{\mu\mu}^{-1}$ which after some algebra leads to the systems

$$\begin{aligned}\epsilon_\mu &= \frac{H\mu}{2}\epsilon_\mu^{(0)} - \sum_{k=1}^{\mu-1} Q_k * (-1)^{r(k)} T \ln\left(1+\exp\left(-\frac{\epsilon_k}{T}\right)\right)\\ &\quad - K * (-1)^{r(\mu)} T \ln\left(1+\exp\left(\frac{\epsilon_\mu}{T}\right)\right)\\ \epsilon_j &= \delta_{j\mu-1}\frac{H\mu}{2}\epsilon_j^{(0)} + \sum_{k=1}^{\mu-1} B_{jk} * (-1)^{r(k)} T \ln\left(1+\exp\left(-\frac{\epsilon_k}{T}\right)\right)\\ &\quad - K * (-1)^{r(\mu)} T \ln\left(1+\exp\left(\frac{\epsilon_\mu}{T}\right)\right)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}-(-1)^{r(\mu)}(\tilde{\rho}_\mu + \rho_\mu) &= \frac{1}{2\pi c}\epsilon_\mu^{(0)} - \sum_{k=1}^{\mu-1} Q_k * \rho_k + K * \tilde{\rho}_\mu\\ -(-1)^{r(j)}(\tilde{\rho}_j + \rho_j) &= \frac{1}{2\pi c}\epsilon_j^{(0)} + \sum_{k=1}^{\mu-1} B_{jk} * \rho_k + Q_j * \rho_\mu\end{aligned} \tag{4.2}$$

where we have introduced

$$\begin{aligned}K(\lambda) &= -T_{\mu\mu} * A_{\mu\mu}^{-1}(\lambda)\\ Q_k(\lambda) &= -T_{\mu k} * A_{\mu\mu}^{-1}(\lambda)\\ B_{jk}(\lambda) &= (T_{jk} + T_{j\mu} * A_{\mu\mu}^{-1} * T_{\mu k})(\lambda).\end{aligned} \tag{4.3}$$

Now we want to perform a shift of the λ-variable in the functions $\epsilon_j(\lambda)$ in the following way:

$$\varphi_j(\lambda) = \frac{1}{T}\epsilon_j\left(\lambda + a\ln\frac{T}{2\pi|c|}\right) \tag{4.4}$$

where the constant a will be determined yet. We choose it in a way that $\varphi_j(\lambda)$ for $\lambda \to \infty$ has a finite limit if $T \to 0$. From paper I (3.24) we can see, that for all $\epsilon_j^{(0)}(\lambda) \sim \exp(\pi\lambda/(\pi-\gamma))$ if $\lambda \to \infty$. Therefore $a = -(\pi-\gamma)/\gamma$.

After that shift the system (4.1) is rewritten as

$$\begin{aligned}\varphi_\mu &= \frac{H\mu}{2T} - \sum_{k=1}^{\mu-1} Q_k * (-1)^{r(k)} \ln(1+\exp(-\varphi_k)) - K * (-1)^{r(\mu)} \ln(1+\exp(\varphi_\mu))\\ &\quad + \frac{1}{\pi-\gamma}\exp\left(-\frac{\pi\lambda}{\pi-\gamma}\right)\left[1 + 2\cos\frac{\pi}{2(\mu-1)}\right]\\ \varphi_j &= \delta_{j\mu-1}\frac{H\mu}{2T} + \sum_{k=1}^{\mu-1} B_{jk} * (-1)^{r(k)} \ln(1+\exp(-\varphi_k))\\ &\quad - K * (-1)^{r(\mu)} \ln(1+\exp(\varphi_\mu))\\ &\quad + \frac{2}{\pi-\gamma}\exp\left(-\frac{\pi\lambda}{\pi-\gamma}\right)\left[\cos\frac{(\mu-j-1)\pi}{2(\mu-1)} + \cos\frac{(\mu-j)\pi}{2(\mu-1)}\right.\\ &\quad \left. + \cos\frac{(\mu-j-1)\pi}{2(\mu-2)}\right].\end{aligned} \tag{4.5}$$

The system (4.2) is treated analogously.

After differentiating equation (4.5) one obtains the important relations

$$\rho_j\left(\lambda-\frac{\mu-1}{\mu}\ln\frac{T}{2\pi|c|}\right)=(-1)^{r(j)}\frac{\pi-\gamma}{2\pi^2|c|}T\frac{\partial}{\partial\lambda}\ln(1+\exp(-\varphi_j))$$
$$\tilde\rho_j\left(\lambda-\frac{\mu-1}{\mu}\ln\frac{T}{2\pi|c|}\right)=-(-1)^{r(j)}\frac{\pi-\gamma}{2\pi^2|c|}T\frac{\partial}{\partial\lambda}\ln(1+\exp(\varphi_j)) \tag{4.6}$$

for $j=1\ldots\mu$.

These relations are necessary to make the appropriate substitutions of variables in the integrals for S and S_z. No such relations are expected as soon as $\bar c\neq\tilde c$.

The starting point for the heat capacity calculation is the expression for the entropy

$$\frac{S}{N}=\sum_{j=1}^{\mu}\int_{-\infty}^{\infty}\mathrm{d}\lambda\left[\rho_j\ln\left(1+\frac{\tilde\rho_j}{\rho_j}\right)+\tilde\rho_j\ln\left(1+\frac{\rho_j}{\tilde\rho_j}\right)\right]. \tag{4.7}$$

Using symmetry and $\tilde\rho_j/\rho_j=\mathrm{e}^{\epsilon_j/T}$ we have

$$\frac{S}{N}=2\sum_{j=1}^{\mu}\int_{0}^{\infty}\mathrm{d}\lambda\,[\rho_j\ln(1+\mathrm{e}^{\epsilon_j/T})+\tilde\rho_j\ln(1+\mathrm{e}^{-\epsilon_j/T})]. \tag{4.8}$$

For $T\to0$ the main contribution to the integral comes from $\lambda\gg1$.

After performing the shift (4.4) and using relations (4.6) the entropy becomes

$$\frac{S}{N}=\frac{\pi-\gamma}{\pi^2|c|}\sum_{j=1}^{\mu}(-1)^{r(j)}\int_{\frac{\mu-1}{\mu}\ln\left(\frac{T}{2\pi|c|}\right)}^{\infty}\mathrm{d}\lambda\left[\frac{\partial}{\partial\lambda}\ln(1+\mathrm{e}^{-\varphi_j})\ln(1+\mathrm{e}^{\varphi_j})\right.$$
$$\left.+\frac{\partial}{\partial\lambda}\ln(1+\mathrm{e}^{\varphi_j})\ln(1+\mathrm{e}^{-\varphi_j})\right]. \tag{4.9}$$

We are interested only in the leading order for vanishing temperature. Therefore we can substitute the lower limit by $-\infty$ (both integrals converge).

We will see in a moment that the remaining integral is even independent of T.

Now it is straightforward to change the variable in the way

$$x=\frac{1}{1+\mathrm{e}^{\varphi_j}}\equiv f(\varphi_j) \tag{4.10}$$

for every integral in the sum (note the change in the definition of the function f in equation (3.18) of paper I).

In final form

$$\frac{S}{N}=-\frac{\pi-\gamma}{\pi^2|c|}\sum_{j=1}^{\mu}(-1)^{r(j)}\int_{f(\varphi_j^-)}^{f(\varphi_j^+)}\mathrm{d}x\left[\frac{\ln x}{1-x}+\frac{\ln(1-x)}{x}\right] \tag{4.11}$$

with $\varphi_j^{\pm}=\varphi_j(\pm\infty)$.

The integral is given by the function $\gamma(a,b)$ already introduced in [9] from where the necessary special values have also been taken.

$$\gamma(a,b)=\int_a^b\mathrm{d}x\left[\frac{\ln x}{1-x}+\frac{1}{x}\ln(1-x)\right]. \tag{4.12}$$

Following the standard procedure [8] it is more convenient to use another form of the TBAE to determine $\varphi_j^{\pm}$. They differ from those of the $XXZ(S)$ model [9] in the terms with the coupling constants only. It is the system (3.17) from paper I.

After the shift it takes the form

$$
\begin{aligned}
\varphi_1(\lambda) &= -s * \ln f(\varphi_2)(\lambda) + |c| \exp\left(-\frac{\pi\lambda}{\pi - \gamma}\right) \\
\varphi_j(\lambda) &= -s * \ln[f(\varphi_{j+1}) f(\varphi_{j-1})](\lambda) + |c| \exp\left(-\frac{\pi\lambda}{\pi - \gamma}\right) \delta_{j2} \\
\varphi_{\mu-1}(\lambda) &= \frac{H\mu}{2T} - s * \ln f(\varphi_{\mu-2})(\lambda) \\
\varphi_\mu(\lambda) &= \frac{H\mu}{2T} + s * \ln f(\varphi_{\mu-2})(\lambda).
\end{aligned}
\tag{4.13}
$$

For $\lambda \to -\infty$ the inhomogeneous terms generate a solution of the form

$$
\begin{aligned}
\varphi_j^- &= +\infty \qquad j = 1 \ldots \mu - 1 \\
\varphi_\mu^- &= -\infty
\end{aligned}
\tag{4.14}
$$

which implies

$$
\begin{aligned}
f(\varphi_j^-) &= 0 \qquad j = 1 \ldots \mu - 1 \\
f(\varphi_\mu^-) &= 1.
\end{aligned}
\tag{4.15}
$$

For $\lambda \to \infty$ the free terms can be neglected, and thus the solution is given in [9]

$$
\begin{aligned}
f(\varphi_j^+) &= \left[\frac{\sinh(H/2T)}{\sinh(H(j+1)/2T)}\right]^2 \\
\varphi_{\mu-1}^+ &= \frac{H\mu}{2T} + \ln\left[\frac{\sinh(H(\mu-1)/2T)}{\sinh(H/2T)}\right] \\
\varphi_\mu^+ &= \frac{H\mu}{2T} - \ln\left[\frac{\sinh(H(\mu-1)/2T)}{\sinh(H/2T)}\right].
\end{aligned}
\tag{4.16}
$$

For $H \to 0$ then

$$
f(\varphi_j^+) = \frac{1}{(j+1)^2} \qquad f(\varphi_{\mu-1}^+) = \frac{1}{\mu} \qquad f(\varphi_\mu^+) = 1 - \frac{1}{\mu}. \tag{4.17}
$$

We mention that the above solution does not depend on the sign of the coupling constant. The consequences of that fact will be considered below.

Now we can calculate relation (4.11):

$$
\begin{aligned}
\frac{S}{N} &= \frac{(\pi-\gamma)T}{\pi^2|c|} \left\{ \sum_{j=1}^{\mu-2} \gamma\left(\frac{1}{(j+1)^2}, 0\right) + \gamma\left(\frac{1}{\mu}, 0\right) - \gamma\left(1 - \frac{1}{\mu}, 1\right) \right\} \\
&= \frac{(\pi-\gamma)T}{\pi^2|c|} \left\{ \sum_{j=1}^{\mu-2} \gamma\left(\frac{1}{(j+1)^2}, 0\right) + 2\gamma\left(\frac{1}{\mu}, 0\right) \right\} \\
&= \frac{(\pi-\gamma)T}{\pi^2|c|} \left\{ \frac{1}{3}\pi^2 \right\} \\
&= \frac{(\pi-\gamma)T}{3|c|}.
\end{aligned}
\tag{4.18}
$$

Finally, for the heat capacity per site (of a chain with $2N$ sites)

$$
C = \frac{\pi - \gamma}{6|c|} T. \tag{4.19}
$$

The polarization is obtained in the same way as in [9] starting with the basic formula

$$\frac{S^z}{N} = \frac{\mu}{2}\left\{\int_{-\infty}^{\infty} \mathrm{d}\lambda\, \tilde{\rho}_{\mu-1}(\lambda) - \int_{-\infty}^{\infty} \mathrm{d}\lambda\, \rho_\mu(\lambda)\right\} \tag{4.20}$$

which for our model takes the same form. Using symmetry and the shift gives

$$\begin{aligned}\frac{S^z}{N} &= 2\frac{\mu}{2}\left\{\int_{0}^{\infty} \mathrm{d}\lambda\, \tilde{\rho}_{\mu-1}(\lambda) - \int_{0}^{\infty} \mathrm{d}\lambda\, \rho_\mu(\lambda)\right\}\\ &= \mu\left\{-\int_{-\infty}^{\infty} \frac{(\pi-\gamma)T}{2\pi^2|c|}\frac{\partial}{\partial\lambda}\ln(1+\mathrm{e}^{\varphi_{\mu-1}})\,\mathrm{d}\lambda\right.\\ &\qquad\left.-\int_{-\infty}^{\infty} \frac{(\pi-\gamma)T}{2\pi^2|c|}\frac{\partial}{\partial\lambda}\ln(1+\mathrm{e}^{-\varphi_{\mu}})\,\mathrm{d}\lambda\right\}\\ &= \frac{\mu(\pi-\gamma)T}{2\pi^2|c|}\left\{\ln\left[\frac{1+\exp(-\varphi_\mu^+)}{1+\exp(\varphi_{\mu-1}^+)}\right] - \ln\left[\frac{1+\exp(-\varphi_\mu^-)}{1+\exp(\varphi_{\mu-1}^-)}\right]\right\}.\end{aligned} \tag{4.21}$$

For the first term we use relations (4.16) with $H \gg T$.

$$\begin{aligned}\varphi_{\mu-1}^+ &= \frac{H\mu}{2T} + \ln\left[\frac{\exp(H(\mu-1)/2T)}{\exp(H/2T)}\right] = \frac{H(\mu-1)}{T}\\ \varphi_\mu^+ &= \frac{H\mu}{2T} - \ln\left[\frac{\exp(H(\mu-1)/2T)}{\exp(H/2T)}\right] = \frac{H}{T}.\end{aligned} \tag{4.22}$$

and hence

$$\begin{aligned}&1+\exp(\varphi_{\mu-1}^+) = \exp\left(\frac{H(\mu-1)}{T}\right)\\ &1+\exp(-\varphi_\mu^+) = 1.\end{aligned} \tag{4.23}$$

In the second term (4.14) must be completed by corrections containing the leading term in H

$$\ln\left[\frac{1+\exp(-\varphi_\mu^-)}{1+\exp(\varphi_{\mu-1}^-)}\right] \cong -\varphi_\mu^- - \varphi_{\mu-1}^- = -\frac{H\mu}{T}. \tag{4.24}$$

Putting things together

$$\frac{S^z}{N} = \mu\frac{(\pi-\gamma)H}{2\pi^2|c|} \tag{4.25}$$

and finally for the susceptibility

$$\chi = \frac{\mu-1}{4\pi|c|} = \frac{\pi-\gamma}{4\pi\gamma|c|}. \tag{4.26}$$

Now we perform the calculation for $\bar{c} = \tilde{c} = c > 0$. We will list only the necessary changes in the calculation, induced by the sign of the coupling constant. For $c > 0$ we have

$$\begin{aligned}&\epsilon_1, \epsilon_2 \leqslant 0\\ &\epsilon_j \geqslant 0 \qquad j = 3\ldots\mu.\end{aligned}$$

The shift in equation (4.4) must now be taken as $a = -\pi/\gamma$ according to the asymptotics of the $\epsilon_j^{(0)}$. Instead of equation (4.6) we have then

$$\begin{aligned}\rho_j\left(\lambda - \mu\ln\frac{T}{2\pi c}\right) &= (-1)^{r(j)}\frac{\gamma}{2\pi^2 c}T\frac{\partial}{\partial\lambda}\ln(1+\exp(-\varphi_j))\\ \tilde{\rho}_j\left(\lambda - \mu\ln\frac{T}{2\pi c}\right) &= -(-1)^{r(j)}\frac{\gamma}{2\pi^2 c}T\frac{\partial}{\partial\lambda}\ln(1+\exp(\varphi_j)).\end{aligned} \tag{4.27}$$

Note the change of the overall sign.

Consequently, (4.11) is modified

$$\frac{S}{N} = -\frac{\gamma}{\pi^2 c}\sum_{j=1}^{\mu}(-1)^{r(j)}\int_{f(\varphi_j^-)}^{f(\varphi_j^+)} \mathrm{d}x \left[\frac{\ln x}{1-x} + \frac{\ln(1-x)}{x}\right]. \tag{4.28}$$

The change in the system (4.13) is obviously the replacement of $|c|$ by $-c$. As already mentioned above, there are no changes in the solutions for $\lambda \to \infty$. For $\lambda \to -\infty$ the solution is contained in [9]:

$$\begin{aligned}
&f(\varphi_1^-) = f(\varphi_2^-) = 1\\
&f(\varphi_j^-) = \left[\frac{\sinh((H\mu/2T)1/(\mu-2))}{\sinh((H\mu/2T)(j-1)/(\mu-2))}\right]^2 \qquad j = 3\ldots\mu-2\\
&\varphi_{\mu-1}^- = \frac{H\mu}{2T} + \ln\left[\frac{\sinh((H\mu/2T)(\mu-3)/(\mu-2))}{\sinh(H\mu/2T)1/(\mu-2)}\right]\\
&\varphi_{\mu}^- = \frac{H\mu}{2T} - \ln\left[\frac{\sinh((H\mu/2T)(\mu-3)/(\mu-2))}{\sinh(H\mu/2T)1/(\mu-2)}\right].
\end{aligned} \tag{4.29}$$

For $H \to 0$ this implies

$$\begin{aligned}
&f(\varphi_j^-) = \frac{1}{(j-1)^2} \qquad j = 3\ldots\mu-2\\
&f(\varphi_{\mu-1}^-) = \frac{1}{\mu-2} \qquad f(\varphi_\mu^-) = 1 - \frac{1}{\mu-2}.
\end{aligned} \tag{4.30}$$

Now we are ready to find the sum in equation (4.18)

$$\begin{aligned}
\sum_{j=1}^{\mu}&(-1)^{r(j)}\gamma(\varphi_j^-,\varphi_j^+) = \gamma(1,\tfrac{1}{4}) + \gamma(1,\tfrac{1}{9})\\
&+\sum_{j=3}^{\mu-2}\gamma\left(\frac{1}{(j-1)^2},\frac{1}{(j+1)^2}\right) + \gamma\left(\frac{1}{\mu-2},\frac{1}{\mu}\right) - \gamma\left(1-\frac{1}{\mu-2},1-\frac{1}{\mu}\right)\\
&= \gamma(1,\tfrac{1}{4}) + \gamma(1,\tfrac{1}{9}t) + \sum_{j=1}^{\mu-4}\gamma\left(\frac{1}{(j+1)^2},0\right)\\
&-\sum_{j=1}^{\mu-2}\gamma\left(0,\frac{1}{(j+1)^2}\right) + \gamma(0,\tfrac{1}{4}) + \gamma(0,\tfrac{1}{9}) + 2\gamma\left(\frac{1}{\mu-2},\frac{1}{\mu}\right)\\
&= 2\gamma(1,0) + \frac{\pi^2}{3} - 2\gamma\left(\frac{1}{\mu-2},0\right) + 2\gamma\left(0,\frac{1}{\mu}\right) - \frac{\pi^2}{3} + 2\gamma\left(\frac{1}{\mu-2},\frac{1}{\mu}\right)\\
&= 2\gamma(1,0)\\
&= \frac{2\pi^2}{3}.
\end{aligned} \tag{4.31}$$

Finally,

$$C = \frac{\gamma T}{3c}. \tag{4.32}$$

Polarization (4.21) is modified

$$\frac{S^z}{N} = \frac{T}{2\pi c}\left\{\ln\left[\frac{1+\exp(\varphi_{\mu-1}^+)}{1+\exp(-\varphi_\mu^+)}\right] - \ln\left[\frac{1+\exp(\varphi_{\mu-1}^-)}{1+\exp(-\varphi_\mu^-)}\right]\right\}. \tag{4.33}$$

Relations (4.23) are still valid. From equation (4.29) we obtain

$$\varphi^-_{\mu-1} = \frac{H\mu}{T}\frac{\mu-3}{\mu-2} \qquad \text{and} \qquad \varphi^-_{\mu} = \frac{H\mu}{T}\frac{1}{\mu-2}. \tag{4.34}$$

Therefore,

$$\frac{S^z}{N} = \frac{T}{2\pi c}\left\{\frac{H}{T}\left[\mu - 1 - \mu + \frac{\mu}{\mu-2}\right]\right\}$$
$$= \frac{T}{2\pi c}\frac{H}{T}\frac{2}{\mu-2} \tag{4.35}$$

and finally

$$\chi = \frac{1}{2\pi c}\frac{1}{\mu-2} = \frac{1}{2\pi c}\frac{\gamma}{\pi-2\gamma}. \tag{4.36}$$

Now we have to compare our results with those of other authors who have presented calculations especially for $c > 0$. To avoid ambiguities we multiply heat capacity and susceptibility by the speed of sound v_s, afterwards the result becomes unique, not depending on the normalization of the coupling constant.

From [1] and [2] we can derive

$$v_s = \frac{2c\pi}{\gamma} \qquad \text{for } c > 0 \qquad \text{and} \qquad v_s = \frac{2|c|\pi}{\pi-\gamma} \qquad \text{for } c < 0. \tag{4.37}$$

We have also used our method to obtain the values for the two homogeneous systems ($s = \frac{1}{2}$ and $s = 1$). At least, the susceptibility for $s = 1$ has not been calculated before in the case of negative coupling. (Heat capacity was determined in paper [10].) In all cases considered we found, for the heat capacity, the conformal result

$$C v_s = \frac{c_v T\pi}{3} \tag{4.38}$$

where c_v is the central charge of the Virasoro algebra, which is equal to one for negative coupling.

It is remarkable that formula (4.11) is preserved, because (rewritten for the entropy per site) the factor in front of the sum is always equal to $1/(v_s\pi)$ (apart from the sign) while the sum measures the central charge (being equal to $\pi^2 c_v/3$).

The form of equation (4.21) for the polarization per site can be understood in the same way. The factor in front of the logarithms is always $T/(2v_s\gamma)$, while for different signs and spins the logarithms also differ.

For positive coupling they yield the result $2S'/(\pi - 2\gamma S')H/T$ where S' is the larger of the two spins (our model has $S' = 1$), which may be equal.

Therefore

$$v_s\chi(c > 0) = \frac{S'}{\pi - 2S'\gamma} \tag{4.39}$$

which is consistent with all former results, especially with paper [9] for homogeneous chains and with paper [4] for the isotropic limit of alternating chains with $S' > S$.

For negative coupling the logarithms always equal H/T leading to

$$v_s\chi(c < 0) = \frac{1}{2\gamma} \tag{4.40}$$

with no dependence on the spins. This is remarkable, because we remember $c_v = 1$ in the same case.

5. The isotropic model with alternating spins

In this section we present some results for the isotropic limit of the model considered above, which we will call $XXX(\frac{1}{2}, 1)$. On one side, there are some peculiarities in the limit $\gamma \to 0$ (especially for negative couplings). On the other side, in the sectors with different signs of couplings it is possible to obtain several new results yet undiscovered for the anisotropic case.

To begin with we have to define the isotropic limit. The model has been considered in [6], we have only to specify the normalization of coupling constants to fit with our section 2.

The BAE take the form

$$\left(\frac{\lambda_j + \frac{i}{2}}{\lambda_j - \frac{i}{2}} \frac{\lambda_j + i}{\lambda_j - i}\right)^N = -\prod_{k=1}^{M} \frac{\lambda_j - \lambda_k + i}{\lambda_j - \lambda_k - i} \qquad j = 1 \ldots M. \tag{5.1}$$

Instead of (2.3) we define

$$E = c_1 E_1 + c_2 E_2 - (\tfrac{3}{2}N - M)H \tag{5.2}$$

with

$$E_i = -\sum_{j=1}^{M} a_i(\lambda_j) \qquad i = 1, 2 \tag{5.3}$$

where

$$a_n(\lambda) = \frac{n}{\lambda^2 + \frac{n^2}{4}}. \tag{5.4}$$

Taking the limit of equations (2.4) and (2.5) we see that we have to put

$$c_1 = \lim_{\gamma \to 0} \frac{\bar{c}}{\gamma} \qquad \text{and} \qquad c_2 = \lim_{\gamma \to 0} \frac{\tilde{c}}{\gamma}. \tag{5.5}$$

The TBA (for zero temperature) has been given in [6]:

$$\begin{aligned} \epsilon_1(\lambda) &= -2\pi c_1 p(\lambda) + p * \epsilon_2^+(\lambda) \\ \epsilon_2(\lambda) &= -2\pi c_2 p(\lambda) + p * \epsilon_1^+(\lambda) + h * \epsilon_2^+(\lambda) + \frac{H}{2}. \end{aligned} \tag{5.6}$$

Here

$$\begin{aligned} p(\lambda) &= \frac{1}{2\cosh \pi\lambda} \\ h(\lambda) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{-|p|/2}}{2\cosh(p/2)} \mathrm{e}^{\mathrm{i}p\lambda}. \end{aligned} \tag{5.7}$$

As in section 3 we must distinguish between the various regions of signs when the system (5.6) is solved. We will follow exactly the notation from section 3 above.

The solutions in the sectors (i) and (iv) are well known. While in (i) the ground state is antiferromagnetic (1- and 2-strings) and therefore the limit of the anisotropic case, in (iv) the ground state is ferromagnetic and hence different from the anisotropic case. This explains why the results of section 3 for $c > 0$ (with the replacement (5.5)) lead to the isotropic values, while they diverge for $c < 0$.

Now we wish to investigate, in some detail, sector (ii) with $c_1 > 0$, $c_2 < 0$. Only real roots can be present in the ground state. The TBA (only one equation left) can be formulated in two equivalent ways; we shall need both.

$$\epsilon_1(\lambda) = -2\pi c_1 p(\lambda) + 2\pi |c_2| h(\lambda) + h * \epsilon_1^+(\lambda) + \frac{H}{2} \tag{5.8}$$

$$\epsilon_1^+(\lambda) = -c_1 a_1(\lambda) + |c_2| a_2(\lambda) - \left[\delta(\lambda) + \frac{a_2(\lambda)}{2\pi}\right] * \epsilon_1^- + H. \tag{5.9}$$

From equation (5.9) one easily determines the region where the solution is ferromagnetic. The integral term is always non-negative. Therefore, we have ferromagnetic behaviour ($\epsilon(\lambda) > 0$ everywhere) if the remaining function of λ on the r.h.s. of equation (5.9) is strictly positive. That is guaranteed, if it is fulfilled for $\lambda = 0$. Thus

$$-4c_1 + 2|c_2| + H > 0 \tag{5.10}$$

implying

$$H_{crit} = 4c_1 - 2|c_2| \tag{5.11}$$

in this region. For a vanishing magnetic field ferromagnetism is obtained as long as

$$0 \leqslant \frac{c_1}{|c_2|} \leqslant \frac{1}{2}. \tag{5.12}$$

This is just the isotropic limit of inequality (3.13). For $c_1 = \frac{1}{2}|c_2|$ there is a phase transition to a partially ordered state, the Fermi zone of the 1-strings starts at $\lambda = 0$. The Fermi radius increases and stays finite for $H \neq 0$ moving counter-clockwise towards the vertical c_1-axis in the (c_2, c_1)-plane. We have strictly proven that there is no point where it reaches infinity (for $H = 0$) unless $c_2 = 0$. One can see that from equation (5.9), because for $\lambda \to \infty$ $h(\lambda)$ vanishes much slower than $p(\lambda)$.

Summarizing the facts, (ii) splits into two parts, one ferromagnetic (5.12) and one with a partially ordered ground state whose Fermi radius varies from zero to infinity (see figure 2). So one can say that the second coupling c_2 works here like an external magnetic field rendering the Fermi radius finite as it is for for the homogeneous antiferromagnetic models with $0 < H < H_{crit}$.

Analytical solutions of equations (5.8) or (5.9) can be obtained for large and small Fermi radius. We start with the first and consider equation (5.8). It is identical to the TBA of $XXX(\frac{1}{2})$ model except for the term with c_2. We therefore use the technique of [11] (see also [12]) recasting that term (after Fourier transformation) as a suitable product. We put as usual $y(\lambda) = \epsilon_1(\lambda + b)$ with $\epsilon_1(b) = y(0) = 0$ and use the symmetry of $\epsilon_1(\lambda)$. After Fourier transformation

$$f(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\lambda} f(\lambda)\,\mathrm{d}\lambda 5.13) \tag{5.13}$$

we write the c_2-term on the r.h.s. of equation (5.8) in the form $-C(\omega)h(\omega)$ with

$$C(\omega) = -2\pi |c_2| \mathrm{e}^{-\mathrm{i}\omega b}. \tag{5.14}$$

With the notations of [12] the solution in ω-space is given by

$$y_+(\omega) = (1 - G_+(\omega))C(\omega) + G_+(\omega)Q_+(\omega). \tag{5.15}$$

This has to be integrated to obtain the radius b. We have carried out the first integral by deforming it along the cut on the negative imaginary axis yielding (for $b \gg 1$)

$$\int_{-\infty}^{\infty} (1 - G_+(\omega))C(\omega)\,\mathrm{d}\omega = \frac{2\pi |c_2| \sqrt{2}}{b^2}. \tag{5.16}$$

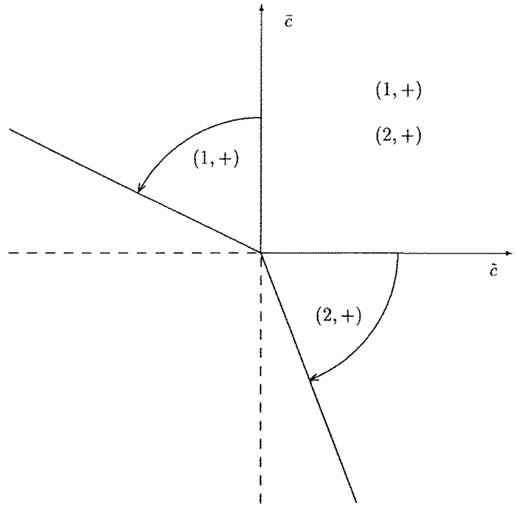

Figure 2. The phase structure of the $XXX(\frac{1}{2}, 1)$ model. The sector without indication of ground-state strings is ferromagnetic. Again axes are drawn broken except if they coincide with sector borders, where they are drawn full.

Together with the second part we have the condition

$$H - c_1 2\pi \sqrt{\frac{2\pi}{e}}\, \mathrm{e}^{-\pi b} + \frac{|c_2| 4}{b^2} = 0 \tag{5.17}$$

which determines $b = b(H, c_1, c_2)$. For its validity we have to ensure

$$H/c_1 \ll 1 \qquad \text{and} \qquad |c_2|/c_1 \ll 1. \tag{5.18}$$

The free energy is calculated using TBA (e.g. see [6]):

$$\frac{F}{2N} = f_0 - \frac{T}{2}\int_{-\infty}^{\infty} p(\lambda)\ln(1 + \mathrm{e}^{\epsilon_1(\lambda)/T})\,\mathrm{d}\lambda - \frac{T}{2}\int_{-\infty}^{\infty} p(\lambda)\ln(1 + \mathrm{e}^{\epsilon_2(\lambda)/T})\,\mathrm{d}\lambda. \tag{5.19}$$

For vanishing temperature the first integral is proportional to $y_+(i\pi)$ yielding (in leading order) the first term in (5.20). The calculation of the second integral is more involved, after one has made use of equation (5.6). We give only the result for the dominant terms:

$$\frac{F}{2N} = f_0' - \frac{1}{8c_1\pi^2}H^2 - \frac{H}{4} - \frac{H}{4\pi b} + \frac{|c_2|}{\pi^2 b^4}. \tag{5.20}$$

The term proportional to H^2 gives $\chi_0 = 1/(4c_1\pi^2)$ which in some sense can be interpreted as half of the value which we found on the conformal line. This is the susceptibility for

$$H \ll |c_2| \ll c_1 \tag{5.21}$$

because then b does not depend on the magnetic field. The term $-H/4$ describes a constant magnetization for $|c_2| \to 0$, which can be found from BAE directly.

The result for $|c_2| \ll H \ll c_1$ is difficult to interpret, the limit $H \to 0$ is not allowed in this case.

Unfortunately we did not succeed in calculating F for $T > 0$, which meets severe difficulties.

We finish the consideration of (ii) by calculating the free energy and magnetic susceptibility for a small Fermi radius, that is close to the line of transition to ferromagnetic behaviour. We solve equation (5.9) for $H < H_{crit}$ with H_{crit} from above (5.11).

Making an expansion in powers of the Fermi radius b we see that the integral on the r.h.s. (except the δ-term) is of power b^3. Therefore, we can easily determine

$$\epsilon_1(\lambda) = \epsilon_1^-(\lambda) \qquad \text{for } |\lambda| \leqslant b \tag{5.22}$$

up to terms of power b^2.

From $\epsilon_1(b) = 0$ we have

$$b = \sqrt{\frac{H_{crit} - H}{16c_1 + 2c_2}} \tag{5.23}$$

and

$$\epsilon_1(\lambda) = (16c_1 + 2c_2)(\lambda^2 - b^2). \tag{5.24}$$

The free energy per site is given by paper I (3.14)

$$\frac{F}{2N} = -\frac{3}{4}H - \frac{2}{\pi}\frac{1}{\sqrt{16c_1 + 2c_2}}(H_{crit} - H)^{3/2} \tag{5.25}$$

and hence

$$\chi = \frac{3}{2\pi}\frac{1}{\sqrt{16c_1 + 2c_2}}\frac{1}{\sqrt{H_{crit} - H}}. \tag{5.26}$$

This is to be compared with the same value for the usual $XXX(\frac{1}{2})$ Heisenberg model

$$\chi_{XXX} = \frac{2}{\pi}\frac{1}{\sqrt{16c}}\frac{1}{\sqrt{H_{crit} - H}} \tag{5.27}$$

in our normalization of coupling constant. That result is, of course, not the limit $c_2 \to 0$ of equation (5.26), because the change in paper I (3.14) must also be taken into account.

At the end of this section we shortly comment on sector (iii). It is treated in the same way as sector (ii). Equations (5.8) and (5.9) are replaced by

$$\epsilon_2(\lambda) = -2\pi c_2 p(\lambda) + \frac{\pi|c_1|\lambda}{\sinh(\pi\lambda)} + \left(\frac{\lambda}{2\sinh(\pi\lambda)} + h(\lambda)\right) * \epsilon_2^- + \frac{H}{2} \tag{5.28}$$

$$\epsilon_2^+(\lambda) = |c_1|a_2(\lambda) - c_2(a_1(\lambda) + a_2(\lambda)) - \left(\delta(\lambda) + \frac{2a_2(\lambda) + a_4(\lambda)}{2\pi}\right) * \epsilon_2^- + 2H. \tag{5.29}$$

The critical magnetic field can be read off from equation (5.9):

$$H_{crit} = \tfrac{8}{3}c_2 - |c_1|. \tag{5.30}$$

For vanishing field we have ferromagnetic behaviour as long as

$$\frac{|c_1|}{c_2} \geqslant \frac{8}{3} \tag{5.31}$$

(compare equation (3.21)).

The power expansion in b is rather simple while the Wiener–Hopf calculation for large b is a little bit more involved; therefore, it is not carried out here. The results will qualitatively agree with those from above.

The phase structure is depicted in figure 2. We have four sectors and two singular lines (the positive axes). Three of the sectors show critical behaviour without mass gap; one of them is truly antiferromagnetic with infinite Fermi zone. The remaining one is ferromagnetic.

6. Conclusions

We have considered the $XXZ(\frac{1}{2}, 1)$ model with strictly alternating spins in one part of its critical region of anisotropy $0 \leqslant \gamma < \pi/2$. We expect a similar but not identical behaviour in the other part $\pi/2 < \gamma < \pi$ because there is no obvious symmetry between the two regions. The model contains two parameters, anisotropy and the ratio of coupling constants, and shows a rich physical structure. Except for the isotropic model (where we have a ferromagnetic region) we found an antiferromagnetic ground state and no mass gap. So the model behaves critically, but it is conformally invariant only on a line $\bar{c} = \tilde{c}$ and also in a large sector including this line and having at least one negative coupling. Around that line there exist sectors where the ground state does not depend on γ, separated from each other by sectors where it depends crucially on γ. It is remarkable that two kinds of sectors are also different with respect to the occurrence of finite Fermi zones.

The sectors around the line with equal couplings are well studied now, their ground states and excitations have been established. At the line $\bar{c} = \tilde{c}$ we have calculated low-temperature heat capacity and magnetic susceptibility. Different signs of couplings cause very different behaviour, e. g. different central charges (1 or 2) and different behaviour of susceptibilities.

A subsequent paper will deal with finite size corrections in those sectors. We expect the standard results in the conformal case, while, apart from that, conformal symmetry does not make any prediction.

The sectors with finite Fermi zones require further treatment, including numerical studies. The same applies to heat capacity and susceptibility for different coupling constants.

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